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APPLICATION OF DIMOVSKI'S CONVOLUTIONAL CALCULUS TO DISTRIBUTED-ORDER TIME-FRACTIONAL DIFFUSION EQUATION ON A BOUNDED DOMAIN

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Abstract. An initial-boundary value problem for the one-dimensional time-fractional diffusion equation of distributed order is considered. Applying the convolutional calculus approach proposed by Dimovski (I.H. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht (1990)), a Duhamel-type representation of the solution is found in the form of a convolution product of a particular solution and the given initial function. A non-classical convolution with respect to the spatial variable is used. The particular solution is found by eigenfunction expansion. Special attention is paid to the study of the time-dependent components in this expansion. It is proven that the obtained solution is a solution in the classical sense. The Duhamel-type representation is used for numerical computation of the solution in some numerical examples.

1. Introduction

Evolution equations containing distributed-order fractional derivatives are extensively studied in the last years. They are used in physics for modeling of various anomalous relaxation and diffusion processes, e.g. diffusion in a multi-fractal medium, see e.g. [18, 19, 21] and the references cited there. The fractional relaxation of distributed order is studied in [13]. The Cauchy problem for the diffusion equation of distributed order is considered in [14, 15] and in [1, 2] this equation is investigated in a distributional setting. Distributed-order calculus as well as different forms of the evolution equation with time-derivative of order distributed over the unit interval are studied in detail in [9]. See also [3] where such equations are considered in an abstract setting.

Distributed-order time-fractional diffusion equations on bounded domains are investigated in [8, 10, 11, 16, 17] where, together with other contributions, useful regularity estimates for the solutions are established.

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The aim of this paper is to apply the convolutional calculus approach of Dimovski [7] to a simple distributed-order time-fractional diffusion equation on a bounded domain in order to obtain a Duhamel-type representation of the solution. This is a compact representation which is convenient for computation and visualization of the solution. It is based on nonclassical convolutions with respect to the spatial variables. The convolutional calculus approach of Dimovski is applicable to initial-boundary value problems with classical boundary conditions, see e.g. [4], as well as nonlocal boundary conditions, see e.g. [5, 6]. Concerning the temporal variable, a wide range of operators can be considered, including higher order or fractional order differential operators, integral or integro-differential operators, etc.

We consider the following initial-boundary value problem for the one-dimensional time-fractional diffusion equation of distributed order

$$\begin{aligned} &\partial_t^{[\mu]} u(x,t) = u_{xx}(x,t), \quad x \in (0,1), \quad t > 0; \\ &u(0,t) = u(1,t) = 0, \quad t \geq 0; \\ &u(x,0) = f(x), \quad x \in [0,1], \end{aligned} \tag{1.1}$$

where $f(.)$ is a given function satisfying

$$f \in C^2([0,1]), \quad f(0) = f(1). \tag{1.2}$$

Here $\partial_t^{[\mu]}$ denotes the distributed order fractional derivative in time, i.e.

$$\partial_t^{[\mu]} = \int_0^1 \mu(\beta) \partial_t^\beta \, d\beta, \tag{1.3}$$

where ∂_t^β is the Caputo fractional derivative

$$\partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} f'(\tau) \, d\tau, \quad \beta \in (0,1), \tag{1.4}$$

and $\mu(.)$ is a given weight function. We suppose that $\mu \in C([0,1])$, $\mu(\beta) \geq 0$, and $\mu(\beta) \not\equiv 0$, $\beta \in [0,1]$.

Developing a bivariate convolutional calculus for problem (1.1) we obtain a Duhamel-type representation of the solution, which is in the form of a convolution product of a particular solution and the given initial function $f(x)$. The particular solution is found through eigenfunction expansion. In order to prove that the obtained solution is a solution in the classical sense, special attention is paid to the representation and estimates for the time-dependent components in the eigenfunction expansion. In this respect we formulate some results which seem to be new. The Duhamel-type representation is convenient for numerical computation and visualization of the solution, which is demonstrated on some examples.

The rest of the paper is organized as follows. In Section 2 a bivariate convolutional calculus related to problem (1.1) is developed, which is based on a non-classical convolution proposed in [7]. In Section 3 the problem is rewritten in algebraic form and the Duhamel-type representation of the solution is derived. Section 4 is devoted to the eigenfunction expansion of the solution, especially to the properties of the time-dependent components. Some numerical examples are considered in Section 5, and plots are presented. Section 6 contains conclusions.

2. Convolutional calculi related to problem (1.1)

To solve the initial-boundary value problem (1.1) we build a bivariate operational calculus of a Mikusiński type, following the general approach proposed by Dimovski in [7]. Central in this approach is the notion of convolution of a linear operator.

Definition 2.1. [7] *Let $L : X \to X$ be a linear operator defined on a linear space X. A bilinear, commutative and associative operation $* : X \times X \to X$ is said to be a convolution of the operator L iff*

$$L(f*g) = (Lf)*g \quad \text{for} \quad \text{any} \quad f, g \in X.$$

If it is given a convolution $*$ in X, then any linear operator $M : X \to X$, satisfying the relation $M(f*g) = (Mf)*g$ for $f, g \in X$, is said to be a multiplier of the convolution algebra $(X, *)$.

Let D be some differential operator and L be its right inverse: $DL = I$. Following [7], we consider also the operator F, which is defined by

$$F = I - LD. \tag{2.1}$$

Note that the identity $DL = I$ implies $FL = 0$ and $F^2 = F$. Therefore F is called defining projector. The following useful properties are proven in [7]

$$D(f*g) = (Df)*g + D((Ff)*g), \quad F(f*g) = F((Ff)*g). \tag{2.2}$$

In what follows we denote the operators acting with respect to the spatial variable with capital letters (e.g. D, L, S) and operators acting with respect to the time variable by lowercase letters (e.g. $\partial^{[\mu]}, l^{[\mu]}, l, s^{[\mu]}$).

2.1. **Spatial variable.** With respect to the spatial variable problem (1.1) contains the square of differentiation, subject to Dirichlet boundary conditions. Let us define in the space $C([0,1])$ the operator L, which is right inverse of the operator $D = d^2/dx^2$ and satisfies $(Lf)(0) = (Lf)(1) = 0$. The operator L is defined explicitly by the identity

$$Lf(x) = \int_0^x (x-\xi)f(\xi)\,d\xi - x\int_0^1 (1-\xi)f(\xi)\,d\xi.$$

Consider the following operation for $f, g \in C([0,1])$

$$\begin{aligned}(f \overset{x}{*} g)(x) = -\frac{1}{2}\int_0^1 \Bigg(&\int_x^\xi f(\xi + x - \eta)g(\eta)\,d\eta \\ &- \int_{-x}^\xi f(|\xi - x - \eta|)g(|\eta|)\operatorname{sgn}((\xi - x - \eta)\eta)\,d\eta\Bigg)\,d\xi.\end{aligned} \tag{2.3}$$

It is proven in [7] that the operation $\overset{x}{*}$ is a convolution of the operator L. Moreover,

$$Lf = \{x\} \overset{x}{*} f. \tag{2.4}$$

From the definition of the defining projector (2.1) we find its explicit representation:

$$Ff(x) = f(0)(1-x) + f(1)x. \tag{2.5}$$

Denote by $\mathcal{M}_x$ the multiplicative set of all multipliers of the convolution algebra $(C_x, \overset{x}{*})$, where $C_x = C([0,1])$. $\mathcal{M}_x$ is a commutative ring and $L \in \mathcal{M}_x$. Moreover, L is a non-divisor of 0. Indeed, $Lf = 0$ implies $DLf = 0$ i.e. $f = 0$. Denote by

$\mathcal{N}_x$ the multiplicative subset of $\mathcal{M}_x$ consisting of all non-zero non-divisors of 0 and consider the multiplier fractions

$$\frac{M}{N},\ M \in \mathcal{M}_x, N \in \mathcal{N}_x$$

with the usual convention

$$\frac{M}{N} = \frac{M_1}{N_1} \text{ iff } MN_1 = M_1 N.$$

Further, we consider numbers, functions, multipliers and multiplier fractions as elements of a single algebraic system: the ring of multiplier fractions. For more detailed description of this procedure we refer to [7], see also the PhD thesis [20].

In the ring of multiplier fractions the operator L can be identified by the function $\{x\}$, since $L = \{x\} \overset{x}{*}$, see identity (2.4). Denote by S the algebraic inverse of the operator L:

$$S = \frac{1}{L}. \tag{2.6}$$

It plays a basic role in the corresponding convolutional calculus, since it can be considered as an "algebraic differentiation operator". More precisely, for $f \in C^2([0,1])$ the following identity holds true

$$f'' = Sf - S\{(1-x)f(0)\} - f(1), \tag{2.7}$$

where $f(1)$ is to be considered as a numerical operator. Indeed, (2.5) implies

$$Lf''(x) = f(x) - Ff(x) = f(x) - f(0)(1-x) - f(1)x.$$

Since $\{x\} \equiv L$, multiplying the last identity by $S = 1/L$ we get (2.7).

2.2. **Temporal variable.** Denote by $\overset{t}{*}$ the classical Duhamel convolution

$$(f \overset{t}{*} g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

The integral operator $l = \int_0^t$ is a multiplier of $\overset{t}{*}$ since $l(f \overset{t}{*} g) = (lf) \overset{t}{*} g$.

With respect to the temporal variable problem (1.1) contains the Caputo differential operator of distributed order (1.3).

Denote by $\mathcal{L}\{f\}$ or $\widehat{f}$ the Laplace transform of the function f:

$$\mathcal{L}\{f\}(p) = \widehat{f}(p) = \int_0^\infty e^{-pt} f(t)\,dt.$$

Consider functions $k_1(t)$ and $k_2(t)$, $t \in \mathbb{R}_+$, such that their Laplace transforms are given by

$$\widehat{k_1}(p) = 1/g(p),\ \widehat{k_2}(p) = g(p)/p, \tag{2.8}$$

where

$$g(p) = \int_0^1 \mu(\beta) p^\beta\,d\beta. \tag{2.9}$$

Since $\widehat{k_1}(p)\widehat{k_2}(p) = 1/p$ these two functions satisfy the convolution property

$$k_1 \overset{t}{*} k_2 \equiv 1. \tag{2.10}$$

By the use of the identity

$$\mathcal{L}\left\{\frac{t^{-\beta}}{\Gamma(1-\beta)}\right\}(p) = p^{\beta-1}, \tag{2.11}$$

we can derive from (2.8) and (2.9) an explicit representation of $k_2(t)$

$$k_2(t) = \int_0^1 \mu(\beta) \frac{t^{-\beta}}{\Gamma(1-\beta)}\, d\beta. \tag{2.12}$$

From (1.3), (1.4) and (2.12) we deduce that the Caputo differential operator of distributed order (1.3) can be rewritten in the following convolution form

$$\partial^{[\mu]} f = k_2 \overset{t}{*} f'. \tag{2.13}$$

The first property in (2.2) applied to the convolution $\overset{t}{*}$ (with $D = d/dt$ and $Ff = f(0)$) gives

$$(f \overset{t}{*} g)' = f' \overset{t}{*} g + f(0)g(t). \tag{2.14}$$

The following alternative representation of $\partial^{[\mu]}$ follows from (2.13) and (2.14)

$$\partial^{[\mu]} f = (k_2 \overset{t}{*} f)' - k_2(t)f(0). \tag{2.15}$$

A function $f(t)$ is said to be in the space $C_{-1}(\mathbb{R}_+)$ if there exists a real number $p > -1$ and $f_1(t) \in C(\mathbb{R}_+)$ such that $f(t) = t^p f_1(t),\ t \in \mathbb{R}_+$.

For $f \in C_{-1}(\mathbb{R}_+)$ define the following convolution operator

$$l^{[\mu]} f = k_1 \overset{t}{*} f. \tag{2.16}$$

Applying (2.13), (2.15), (2.16) and (2.10) it follows

$$\partial^{[\mu]} l^{[\mu]} f = (k_2 * (k_1 \overset{t}{*} f))' - k_2(t)(k_1 \overset{t}{*} f)(0) = ((k_2 \overset{t}{*} k_1) \overset{t}{*} f)' = (1 \overset{t}{*} f)' = f,$$
$$l^{[\mu]} \partial^{[\mu]} f = k_1 \overset{t}{*} (k_2 \overset{t}{*} f') = (k_1 \overset{t}{*} k_2) \overset{t}{*} f' = 1 \overset{t}{*} f' = f - f(0).$$

Summarizing

$$\partial^{[\mu]} l^{[\mu]} f = f, \quad l^{[\mu]} \partial^{[\mu]} f = f - f(0), \tag{2.17}$$

i.e. operators $\partial^{[\mu]}$ and $l^{[\mu]}$ are related in the same way as the classical first order differentiation and integration operators. Therefore, we can call $l^{[\mu]}$ integration operator of distributed order.

For more details on the distributed order calculus we refer to [9], see also the recent paper [12] for a generalization of this calculus.

Now we are ready to repeat the procedure from the previous subsection. The integration operators $l, l^{[\mu]}$ are multipliers of the convolution algebra $(C_{-1}(\mathbb{R}_+), \overset{t}{*})$ and they are non-divisors of 0. Define the ring of multiplier fractions, corresponding to the convolution $\overset{t}{*}$. According to (2.16), the operator $l^{[\mu]}$ can be identified with the function $k_1(t)$ in the ring of multiplier fractions and the classical integration operator l is identified with the constant function $\{1\}$. In the ring of multiplier fractions we define the algebraic inverse of $l^{[\mu]}$:

$$s^{[\mu]} = \frac{1}{l^{[\mu]}}. \tag{2.18}$$

Applying $s^{[\mu]}$ to the second identity in (2.17) we obtain the following relation:

$$\partial^{[\mu]} f = s^{[\mu]} f - f(0)s^{[\mu]}. \tag{2.19}$$

2.3. **Bivariate operational calculus.** Let $\Delta = [0,1]\times\mathbb{R}_+$. By $C_{x,t}(\Delta)$ we denote the space of functions $u(x,t)$, continuous with respect to $x \in [0,1]$, which belong to $C_{-1}(\mathbb{R}_+)$ with respect to $t \in \mathbb{R}_+$.

Based on the convolutions $\overset{x}{*}$ and $\overset{t}{*}$ we define a bivariate convolution $\overset{x,t}{*}$ of two functions $f(x,t)$ and $g(x,t)$ in the space $C_{x,t}(\Delta)$ as follows:

$$(f \overset{x,t}{*} g)(x,t) = \int_0^t f(x,t-\tau) \overset{x}{*} g(x,\tau)\, d\tau. \tag{2.20}$$

The bivariate convolution obeys the following separability property: if

$$f(x,t) = f_1(x)f_2(t), \quad g(x,t) = g_1(x)g_2(t),$$

then

$$\left(f \overset{x,t}{*} g\right)(x,t) = \left(f_1 \overset{x}{*} g_1\right)(x)\left(f_2 \overset{t}{*} g_2\right)(t).$$

Identities (2.4) and (2.16) imply that $\overset{x,t}{*}$ is a convolution of both $l^{[\mu]}$ and L and the relation holds true

$$Ll^{[\mu]}u(x,t) = \{xk_1(t)\} \overset{x,t}{*} u(x,t), \quad u \in C_{x,t}(\Delta). \tag{2.21}$$

We now repeat the construction from Subsection 2.1. Let $\mathcal{M}$ be the set of all multipliers of the convolution algebra $(C_{x,t}(\Delta), \overset{x,t}{*})$. It is a commutative ring. The multipliers L and $l^{[\mu]}$ of $\overset{x,t}{*}$ are examples of elements of $\mathcal{M}$ which are non-zero non-divisors of 0. Let $\mathcal{N} \subset \mathcal{M}$ be the set of all non-zero non-divisors of 0 in $\mathcal{M}$ and let $\mathcal{N}^{-1}\mathcal{M}$ be the ring of multiplier fractions. Define operators $S, s^{[\mu]} \in \mathcal{N}^{-1}\mathcal{M}$ by (2.6) and (2.18), respectively. We consider numbers, functions, multipliers and multiplier fractions as elements of a single algebraic system: the ring of multiplier fractions $\mathcal{N}^{-1}\mathcal{M}$.

From (2.7) and (2.19) we obtain the following properties necessary for the algebraization of problem (1.1):

$$u_{xx} = Su - S\{(1-x)u(0,t)\} - [u(1,t)]_x, \tag{2.22}$$

$$\partial_t^{[\mu]}u = s^{[\mu]}u - s^{[\mu]}l[u(x,0)]_t, \tag{2.23}$$

where $[.]_x$ and $[.]_t$ denote numerical operators with respect to x and t, i.e.

$$[f(t)]_x u(x,t) = f(t) \overset{t}{*} u(x,t), \quad [g(x)]_t u(x,t) = g(x) \overset{x}{*} u(x,t).$$

The following identities, implied by (2.4), (2.16) and (2.21), are satisfied in the ring of multiplier fractions:

$$L \equiv [x]_t,\ L^2 = L[x]_t \equiv \left[\frac{x^3-x}{6}\right]_t,\ l^{[\mu]} \equiv [k_1(t)]_x,\ l \equiv [1]_x, \tag{2.24}$$

$$Ll \equiv \{x\},\ L^2 l \equiv \left\{\frac{x^3-x}{6}\right\},\ Ll^{[\mu]} \equiv \{xk_1(t)\},\ \text{etc.} \tag{2.25}$$

3. Duhamel-type representation of the solution

Based on the developed bivariate operational calculus, in this section we find Duhamel-type representation of the solution of problem (1.1). Applying identities (2.22) and 2.23) we rewrite the problem as a single algebraic equation in the ring of multiplier fractions:

$$s^{[\mu]}u - s^{[\mu]}l[f(x)]_t = Su$$

which algebraic solution is

$$u = \frac{s^{[\mu]} l}{s^{[\mu]} - S} [f(x)]_t. \tag{3.1}$$

The expression in (3.1) is well defined provided $s^{[\mu]} - S$ is a non-divisor of 0 in the ring of multiplier fractions. To prove this we suppose the opposite, i.e. $(s^{[\mu]} - S)U = 0$ for a nonzero $U \in \mathcal{N}^{-1}\mathcal{M}$. Each element $U \in \mathcal{N}^{-1}\mathcal{M}$ can be represented as $U = P/Q$, where $P \in \mathcal{M}$ and $Q \in \mathcal{N}$. Then, by applying e.g. the first identity in (2.25) it follows $U = (PLl)/(QLl) = (P\{x\})/(Q\{x\})$, i.e. $U = u/v$ with $u, v \in C_{x,t}(\Delta)$. Hence, $(s^{[\mu]} - S)u = 0$ for some function $u \in C_{x,t}(\Delta)$, $u \neq 0$. Applying $Ll^{[\mu]}$ it follows

$$(L - l^{[\mu]})u = 0. \tag{3.2}$$

Application to (3.2) of the operators $D\partial^{[\mu]}$ and F (with using that $FL = 0$) gives that $u(x,t)$ is a solution of problem (1.1) satisfying homogeneous boundary conditions. Taking $t = 0$ in (3.2) with using that $(l^{[\mu]}u)(x, 0) = 0$ we deduce that $u(x, 0) = 0$. Therefore, the uniqueness of solution of problem (1.1) (see next section) implies $u \equiv 0$, and thus $U \equiv 0$, which is a contradiction. Hence, $s^{[\mu]} - S$ is a non-divisor of 0.

To interpret the algebraic solution (3.1) as an element of $C_{x,t}(\Delta)$, we set

$$\Omega := \frac{s^{[\mu]} l}{(s^{[\mu]} - S)S^2}. \tag{3.3}$$

Applying (2.6) and the second identity in (2.24), we obtain from (3.3) the following representation of the element Ω

$$\Omega = \frac{s^{[\mu]} l L^2}{s^{[\mu]} - S} = \frac{s^{[\mu]} l}{s^{[\mu]} - S} \left[\frac{x^3 - x}{6} \right]_t. \tag{3.4}$$

By a comparison of (3.4) and (3.1), it follows that the element Ω can be interpreted as a function $\Omega(x,t)$ which is a solution of problem (1.1) with the special initial condition $\Omega(x, 0) = f(x) = (x^3 - x)/6$. Rewriting (3.1) as:

$$u = S^2 \left(\frac{s^{[\mu]} l}{(s^{[\mu]} - S)S^2} \right) [f(x)]_t,$$

we deduce by virtue of (3.3), (2.2), (2.22) the identity

$$u = S^2(\Omega \overset{x}{*} f(x)) = D^2(\Omega \overset{x}{*} f(x)) = D(\Omega \overset{x}{*} (Df)) + D^2(\Omega \overset{x}{*} Ff(x)).$$

Since $Ff = 0$ (which follows from the compatibility condition (1.2)) we conclude from the above identity that

$$u = D(\Omega \overset{x}{*} f''). \tag{3.5}$$

In this way we obtained a representation of the algebraic solution (3.1) as a function.

It remains to simplify this expression, noting that it contains differentiation of the convolution $\overset{x}{*}$ which is an integration operation. Denote

$$f \overset{x}{\tilde{*}} g := D(f \overset{x}{*} g). \tag{3.6}$$

Theorem 3.1. *The operation*

$$(f \overset{x}{\tilde{*}} g)(x) = -\frac{1}{2}\frac{d}{dx}\left(\int_x^1 f(1+x-\eta)g(\eta)\,d\eta \right.$$
$$\left. + \int_{-x}^1 f(|1-x-\eta|)g(|\eta|)\mathrm{sgn}((1-x-\eta)\eta)\,d\eta\right) \tag{3.7}$$

is a convolution of the operator L in $C^1([0,1])$ such that the representation $Lf = \{Lx\} \overset{x}{\tilde{*}} f$ holds. Moreover, for $m,n \in \mathbb{N}$

$$\sin(n\pi x) \overset{x}{\tilde{*}} \sin(m\pi x) = \begin{cases} 0, & m \neq n, \\ (-1)^{n-1}\dfrac{n\pi}{2}\sin(n\pi x), & m = n. \end{cases} \tag{3.8}$$

Proof. The proof is a matter of a direct check and is based on the properties of the original convolution (2.3). □

Using the new convolution, the following Duhamel-type representation of the solution is obtained from (3.5).

Theorem 3.2. *Assume that conditions (1.2) are satisfied. Then the solution of problem (1.1) admits the representation:*

$$u(x,t) = \Omega(x,t) \overset{x}{\tilde{*}} f''(x), \tag{3.9}$$

where $\Omega(x,t)$ is a particular solution of (1.1) with $f(x) = (x^3-x)/6$; (3.9) is a solution in the classical sense.

The strict proof of this theorem will be given at the end of the next section.

4. Eigenfunction expansion of the solution

We begin this section with some known facts, which are explained briefly here for completeness. For more details or generalizations we refer to [8, 9, 10, 11, 13, 16, 17].

Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\phi_n(x)\}_{n\in\mathbb{N}}$ be the Dirichlet eigenvalues and eigenfunctions of the operator $-d^2/dx^2$ on $[0,1]$:

$$\lambda_n = n^2\pi^2, \quad \phi_n(x) = \sqrt{2}\sin(n\pi x).$$

Applying eigenfunction decomposition, the solution of problem (1.1) can be written formally as the series $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$, where the functions $u_n(t)$ satisfy the following ordinary differential equation

$$\partial_t^{[\mu]} u_n(t) = -\lambda_n u_n(t), \quad u_n(0) = f_n. \tag{4.1}$$

Here $f_n = (f, \varphi_n)$ with $(.,.)$ being the inner product in $L^2(0,1)$. By applying Laplace transform and using the property of the Caputo fractional derivative

$$\mathcal{L}\{\partial^\beta f\}(p) = p^\beta \widehat{f}(p) - p^{\beta-1}f(0), \quad \beta \in (0,1),$$

we obtain the formal eigenfunction expansion of the solution:

$$u(x,t) = \sum_{n=1}^{\infty} f_n S_n(t)\varphi_n(x). \tag{4.2}$$

Here the functions $S_n(t)$ are defined by their Laplace transforms

$$\widehat{S_n}(p) = \frac{g(p)}{p(g(p)+\lambda_n)} \tag{4.3}$$

with function $g(.)$ given in (2.9). Note that the uniqueness property of the eigenfunction expansion implies the uniqueness of solution of (1.1).

We observe that from (4.3) and the property of the Laplace transform

$$f(0) = \lim_{p\to+\infty} p\widehat{f}(p),$$

it follows

$$S_n(0) = 1. \tag{4.4}$$

Some other properties of the time-dependent components $S_n(t)$ are given next.

Recall that a function $f(t)$ is said to be completely monotone for $t \geq 0$ iff

$$(-1)^n f^{(n)}(t) \geq 0, \quad \text{for all} \quad n = 0, 1, ...$$

The class of completely monotone functions is denoted by $\mathcal{CMF}$.

Theorem 4.1. *The functions $S_n(t)$, $n \in \mathbb{N}$, are completely monotone functions for $t \geq 0$ and admit the representations*

$$S_n(t) = \int_0^\infty e^{-rt} K_n(r)\, dr, \tag{4.5}$$

where

$$K_n(r) = \frac{1}{\pi r} \frac{\lambda_n B(r)}{(A(r) + \lambda_n)^2 + (B(r))^2} \tag{4.6}$$

with $A(r)$ and $B(r)$ defined by

$$A(r) = \int_0^1 \mu(\beta) r^\beta \cos(\beta\pi)\, d\beta, \;\; B(r) = \int_0^1 \mu(\beta) r^\beta \sin(\beta\pi)\, d\beta.$$

Proof. Taking the inverse Laplace transform of $\widehat{S_n}(p)$ it follows

$$S_n(t) = \frac{1}{2\pi i} \int_{Br} e^{pt} \frac{g(p)}{p(g(p) + \lambda_n)}\, dp, \tag{4.7}$$

where $Br = \{p;\ \Re p = \sigma,\ \sigma > 0\}$ is the Bromwich path. The function $\widehat{S_n}(p)$ has a branch point 0, so we cut off the negative part of the real axis. Note that the functions $g(p) + \lambda_n$ have no zeros in the main sheet of the Riemann surface including its boundaries on the cut. Indeed, if $p = \varrho e^{i\theta}$, with $\varrho > 0$, $\theta \in (-\pi, \pi)$, then $\Im\{g(p) + \lambda_n\} \neq 0$, since $\sin(\beta\theta)$ have the same sign for all $\beta \in (0, 1)$.

We use the Titchmarsh theorem for the Laplace inversion. Bending the Bromwich path in (4.7) into the Hankel path $Ha(\varepsilon)$, which starts from $-\infty$ along the lower side of the negative real axis, encircles the disc $|s| = \varepsilon$ counterclockwise and ends at $-\infty$ along the upper side of the negative real axis, we obtain (4.5), where

$$K_n(r) = -\frac{1}{\pi} \Im \left\{ \left. \frac{g(p)}{p(g(p) + \lambda_n)} \right|_{p = re^{i\pi}} \right\},$$

which gives (4.6).

The complete monotonicity of the functions $S_n(t)$ follows from (4.5) and the positivity of the kernels $K_n(r)$ in view of the Bernstein theorem. $\square$

Representation (4.5) will be used in our numerical experiments.

The complete monotonicity of $S_n(t)$ for $t \geq 0$ together with (4.4) imply directly

$$0 \leq S_n(t) \leq 1, \quad t \geq 0. \tag{4.8}$$

An alternative integral representation of $S_n(t)$ is given next. Although this representation is a particular case of the so-called subordination principle (see e.g. [3], Theorem 5.1.) it seems that it has not been formulated or used in the existing literature.

Theorem 4.2. *The functions $S_n(t)$, $n \in \mathbb{N}$, admit the representations*

$$S_n(t) = \int_0^\infty e^{-\lambda_n \tau} \varphi(t,\tau)\, d\tau \tag{4.9}$$

with function $\varphi(t,\tau)$ defined by

$$\varphi(t,\tau) = \frac{1}{2\pi i} \int_{Br} e^{pt-\tau g(p)} \frac{g(p)}{p}\, dp, \quad t,\tau > 0; \tag{4.10}$$

$\varphi(t,\tau)$ is a probability density function in τ, i.e. it satisfies the properties

$$\varphi(t,\tau) \geq 0, \quad \int_0^\infty \varphi(t,\tau)\, d\tau = 1. \tag{4.11}$$

Proof. It is proven in [9], Subsection 4.2., that the integral in (4.10) is well defined. Let $\varphi(t,\tau)$ be a function defined by (4.10). Therefore the Laplace transform $\widehat{\varphi}(p,\tau)$ of this function with respect to t

$$\widehat{\varphi}(p,\tau) = \int_0^\infty e^{-pt} \varphi(t,\tau)\, dt, \quad p,\tau > 0,$$

is given by

$$\widehat{\varphi}(p,\tau) = \frac{g(p)}{p} e^{-\tau g(p)}, \quad p,\tau > 0, \tag{4.12}$$

Define a function $S(t)$ by (4.9). Application of the Laplace transform gives by using (4.12)

$$\widehat{S}(p) = \int_0^\infty e^{-\lambda_n \tau} \widehat{\varphi}(p,\tau)\, d\tau = \frac{g(p)}{p} \int_0^\infty e^{-\tau(g(p)+\lambda_n)}\, d\tau = \frac{g(p)}{p(g(p)+\lambda_n)}.$$

Comparing this result to (4.3), it follows by the uniqueness of the Laplace transform that $S(t) \equiv S_n(t)$.

The nonnegativity of $\varphi(t,\tau)$ follows from the fact that $\widehat{\varphi}(p,\tau)$ is a completely monotone function in p as a product of two completely monotone functions ($g(p)/p \in \mathcal{CMF}$ and $e^{-\tau g(p)} \in \mathcal{CMF}$) by applying the Bernstein theorem.

The integral identity in (4.11) can be obtained directly by integration of (4.10). □

Theorem 4.2 can be used for obtaining some properties of the functions $S_n(t)$. For example, the already established (4.8) as well as the inequality

$$S_n(t) < S_1(t), \quad t > 0, \quad n = 2, 3, ..., \tag{4.13}$$

follow directly from this theorem.

To prove that the eigenfunction expansion (4.2) is a solution in the classical sense we need an estimate for $\lambda_n S_n(t)$.

Theorem 4.3. *The functions $S_n(t)$, $n \in \mathbb{N}$, obey the estimate:*

$$\lambda_n S_n(t) \leq C \frac{t-1}{t \log t}, \quad t > 0, \tag{4.14}$$

where the constant C does not depend on n and t.

Proof. Denote by Σ_θ the sector

$$\Sigma_\theta := \{p \in \mathbb{C};\ p \neq 0, |\arg p| < \theta\}.$$

For $\rho > 0$ and $\theta \in (0, \pi)$ denote by $\Gamma_{\rho,\theta}$ the contour

$$\Gamma_{\rho,\theta} := \left\{re^{-\mathrm{i}\theta} :\ r \geq \rho\right\} \cup \left\{\rho e^{i\psi} :\ |\psi| \leq \theta\right\} \cup \left\{re^{\mathrm{i}\theta} :\ r \geq \rho\right\},$$

which is oriented counterclockwise.

Let $p \in \overline{\Sigma_\varphi}$, where $\varphi \in (\pi/2, \pi - \delta)$. Then $g(p) \in \overline{\Sigma_\varphi}$, thus $g(s) = re^{\mathrm{i}\theta}$, $r > 0$, $|\theta| < \pi - \delta$. Therefore

$$\left|\frac{g(p)}{g(p) + \lambda_n}\right|^2 = \frac{r^2}{r^2 + 2\lambda_n r \cos\theta + \lambda_n^2} \leq \frac{1}{\sin^2\theta},$$

which implies

$$\left|\frac{g(p)}{g(p) + \lambda_n}\right| \leq C, \tag{4.15}$$

with $C = (\sin\delta)^{-1}$. This together with (4.3) implies

$$|\lambda_n \widehat{S}_n(p)| = \left|g(p)\left(\frac{1}{p} - \widehat{S}_n(p)\right)\right| \leq C\left|\frac{g(p)}{p}\right| \leq C\int_0^1 \mu(\beta)|p|^{\beta-1}\, d\beta.$$

Let $t > 0$. We bend the Bromwich path in (4.7) into the contour $\Gamma := \Gamma_{1/t,\varphi}$, and applying the above estimate obtain

$$\begin{aligned}
|\lambda_n S_n(t)| &\leq C\int_\Gamma e^{\Re(p)t}\left(\int_0^1 \mu(\beta)|p|^{\beta-1}\, d\beta\right)|dp|,\\
&\leq C\left(\int_{1/t}^\infty e^{rt\cos\varphi}\left(\int_0^1 \mu(\beta)r^{\beta-1}\, d\beta\right)dr\right.\\
&\qquad\left. + \int_0^\varphi e^{\cos\psi}\frac{1}{t}\left(\int_0^1 \mu(\beta)t^{1-\beta}\, d\beta\right)d\psi\right).
\end{aligned}$$

Interchanging the order of integration it follows

$$|\lambda_n S_n(t)| \leq C\int_0^1 \mu(\beta)t^{-\beta}\, d\beta \leq C\|\mu\|_{C([0,1])}\int_0^1 t^{-\beta}\, d\beta,$$

which implies (4.14). □

Estimate (4.14) is in agreement with the asymptotic estimates for large t established previously, see e.g. [9], [11], [13].

Estimates (4.8) and (4.14) are used in the next theorem to prove that the formal series expansion (4.2) is convergent and gives the unique classical solution of problem (1.1).

Theorem 4.4. *Assume that conditions (1.2) are satisfied. Then the function $u(x,t)$ defined by the series (4.2) is a classical solution of problem (1.1).*

Proof. The eigenfunctions $\phi_n(x)$ are bounded functions on $[0,1]$. In view of (1.2) after integration by parts in the identity $f_n = (f, \phi_n)$ it follows $|f_n| \leq Cn^{-2}$. This together with the estimate (4.8) implies that the series (4.2) is uniformly convergent on $\overline{\Delta}$, hence $u \in C(\overline{\Delta})$. If $t \geq \varepsilon > 0$, $x \in [0,1]$, then estimate (4.14) implies that the series $\sum f_n S_n(t)\phi_n''(x)$ is uniformly convergent. Hence, termwise differentiation is legitimate and the sum of the series is a continuous function. Therefore (4.2) is a classical solution of problem (1.1). □

Inserting the Fourier coefficients of the function $(x^3 - x)/6$ in (4.2) we obtain the series representation of the function $\Omega(x,t)$ from Theorem 3.2.

Corollary 4.5. *The function $\Omega(x,t)$ admits the following eigenfunction expansion:*

$$\Omega(x,t) = \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} S_n(t) \sin(n\pi x). \tag{4.16}$$

It remains to prove Theorem 3.2. Applying analogous argument as in the proof of Theorem 4.4 it is clear that the series (4.16) is uniformly convergent on $\overline{\Delta}$ after applying termwise $\partial/\partial x$. Therefore $\Omega_x(x,t)$ is a continuous functions on $\overline{\Delta}$ and thus the convolution product in (3.9) is well defined. Inserting the eigenfunction expansion (4.16) of $\Omega(x,t)$ in (3.9) and using the properties (2.14), (3.8), the formula for the Fourier coefficients f_n and the separability property of the bivariate convolution, we obtain (4.2). Therefore (3.9) is a solution of (1.1) in the classical sense.

5. Numerical results

For the numerical examples in this section we consider the case of uniformly distributed order over the interval $[0,1]$: $\mu(\beta) \equiv 1$. In this case the function $g(p)$ from (2.9) has the explicit form

$$g(p) = \frac{p-1}{\log p}$$

and the functions $A(r)$ and $B(r)$ in Theorem 4.1 are given by

$$A(r) = -\frac{(r+1)\log r}{\log^2 r + \pi^2}, \quad B(r) = \frac{\pi(r+1)}{\log^2 r + \pi^2}.$$

For the computation of the time-dependent components $S_n(t)$ their integral representations (4.5) are employed. Plots of the first five eigenmodes $S_n(t)$ are shown in Fig. 1, where we recognize properties (4.8) and (4.13) established in the previous section.

The particular solution $\Omega(x,t)$ is then computed by the use of its eigenfunction expansion (4.16). In Fig. 2. the function $\Omega(x,t)$ is plotted.

The solution of problem (1.1) is then calculated for two different initial functions $f(x)$ by applying the Duhamel-type representation (3.9).

First, the solution for $f(x) = \sin(\pi x)$ computed using the Duhamel-type representation (3.9) is compared to the exact solution for this case $u_{\text{exact}}(x,t) = S_1(t)\sin(\pi x)$. In this way the correctness of formula (3.9) is confirmed.

Second, we take the following initial function $f(x)$ satisfying (1.2):

$$f(x) = x(1-x^2)(\sin(2\pi x) + 1.5) \tag{5.1}$$

and compute the solution $u(x,t)$ of problem (1.1) by applying the Duhamel-type representation (3.9). This solution is visualized in Figures 3 and 4. In Fig. 3(a) we observe as expected ultra-slow decay for $t > t_0$, and in Fig. 3(b) the smoothing effect of the diffusion.

6. Conclusions

Applying the convolutional calculus approach of Dimovski, we found a Duhamel-type representation of the solution of an initial-boundary value problem for the one-dimensional time-fractional diffusion equation of distributed order with Dirichlet boundary conditions. This representation is compact and suitable for numerical implementation.

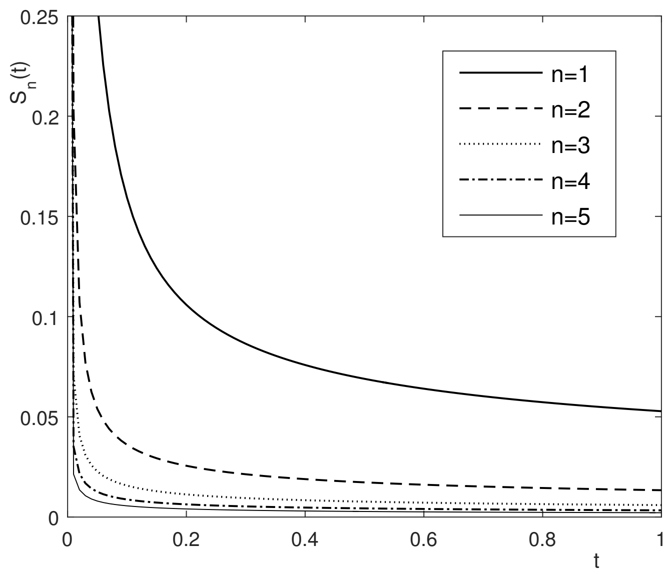


FIGURE 1. Plots of the eigenmodes $S_n(t)$ for $\mu \equiv 1$, $n = 1, 2, ..., 5$.

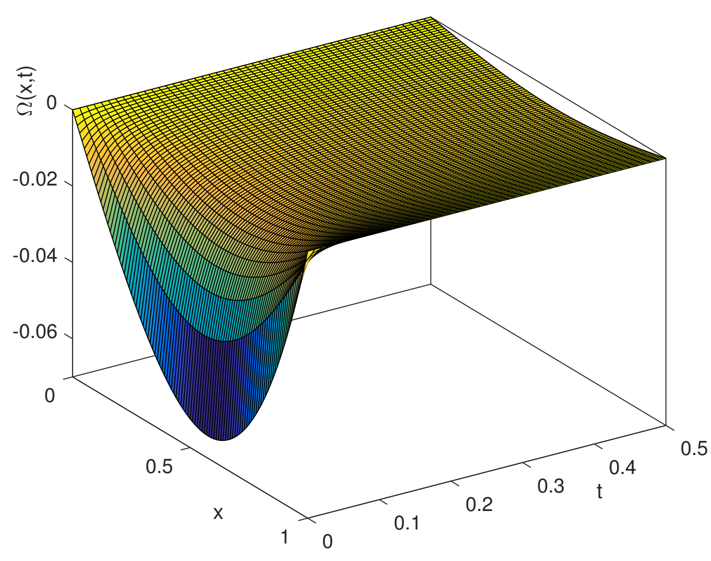
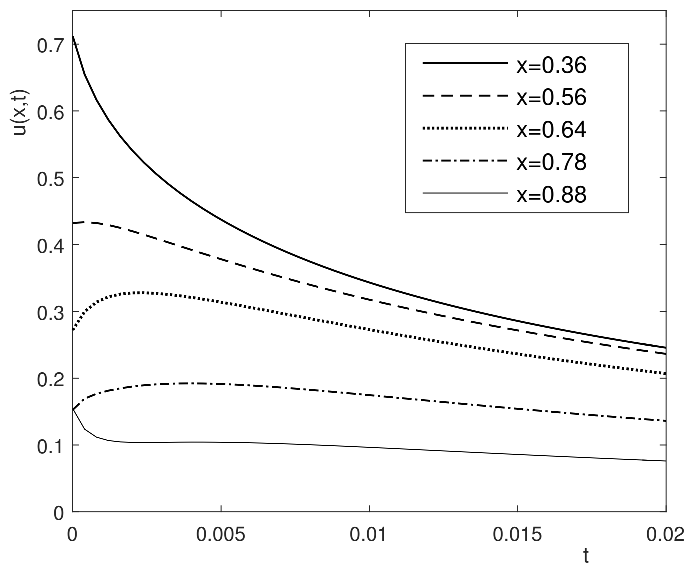
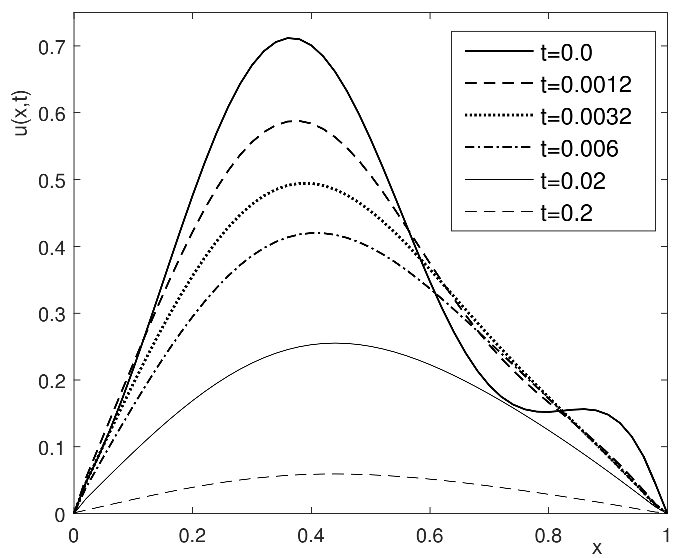


FIGURE 2. The function $\Omega(x, t)$ for $\mu \equiv 1$.

The method used here can be applied to various initial-boundary value problems, subject to a wide range of boundary conditions: of Dirichlet, Neumann or Robin type, as one of the conditions can be also nonlocal. Feasible generalizations include



(a)



(b)

FIGURE 3. Solution $u(x,t)$ of problem (1.1) with $f(x)$ given by (5.1) and $\mu \equiv 1$, plotted as a function of t (a) and x (b).

problems in multidimensional rectangular space domains which can be treated by developing multidimensional operational calculi.

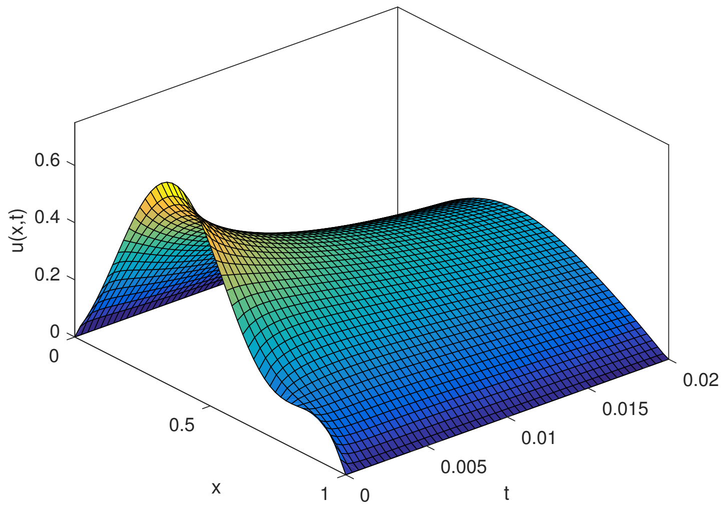


FIGURE 4. Solution $u(x,t)$ of problem (1.1) with $f(x)$ given by (5.1) and $\mu \equiv 1$.

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